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url: <http://www.ijpam.eu>doi: <http://dx.doi.org/10.12732/ijpam.v87i3.4>**ON 3-DIMENSIONAL  $(\epsilon, \delta)$ -TRANS-SASAKIAN STRUCTURE**Y.B. Maralabhavi<sup>1 §</sup>, G.S. Shivaprasanna<sup>2</sup>, G. Somashekhara<sup>3</sup><sup>1,2,3</sup>Department of Mathematics

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**Abstract:** The object of present paper is to study 3-dimensional  $(\epsilon, \delta)$ -trans-Sasakian manifold admitting Ricci solitons and  $K$ -torse forming vector fields. We prove the conditions for the Ricci solitons to be shrinking, expanding and steady. Further, we have obtained a condition for the vector field  $\xi$  in a generalized recurrent 3-dimensional  $(\epsilon, \delta)$ -trans-Sasakian manifold to be co-symplectic. We have also shown that such a manifold  $M$  is of constant scalar curvature.

**AMS Subject Classification:** 53D10, 53D15**Key Words:**  $(\epsilon, \delta)$ -trans-Sasakian manifold, Ricci soliton, shrinking, steady, expanding,  $K$ -torseforming vector field**1. Introduction**

The concept of  $(\epsilon)$ -Sasakian manifolds was introduced by A. Bejancu and K.L. Duggal [1] and further investigation was taken up by Xufend and Xiaoli [13] and Rakesh kumar et al [7]. De and Sarkar [3] introduced and studied conformally flat, Weyl semisymmetric,  $\varphi$ -recurrent  $(\epsilon)$ -Kenmotsu manifolds. In [1], the authors obtained Riemannian curvature tensor of  $(\epsilon)$ -Sasakian manifolds and established relations among different curvatures. H.G. Nagraja et al [4] have studied  $(\epsilon, \delta)$ -trans-Sasakian structures which generalizes both  $(\epsilon)$ -manifolds and  $(\epsilon)$ -Kenmotsu manifolds.

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## 2. Preliminaries

Let  $(M, g)$  be an almost contact metric manifold of dimension  $n$  equipped with an almost contact metric structure  $(\varphi, \xi, \eta, g)$  consisting of a  $(1,1)$  tensor field  $\varphi$ , a vector field  $\xi$ , a 1-form  $\eta$  and a Riemannian metric  $g$  satisfying

$$\varphi^2 = -I + \eta \otimes \xi, \quad (2.1)$$

$$\eta(\xi) = 1, \quad (2.2)$$

$$\varphi\xi = 0, \eta \circ \varphi = 0. \quad (2.3)$$

An almost contact metric manifold  $M$  is called an  $(\epsilon)$ -almost contact metric manifold if

$$g(\xi, \xi) = \epsilon, \quad (2.4)$$

$$\eta(X) = \epsilon g(X, \xi), \quad (2.5)$$

$$g(\phi X, \phi Y) = g(X, Y) - \epsilon \eta(X)\eta(Y), \forall X, Y \in TM, \quad (2.6)$$

where  $\epsilon = g(\xi, \xi) = \pm 1$ . An  $(\epsilon)$ -almost contact metric manifold is called an  $(\epsilon, \delta)$ -trans-Sasakian manifold if

$$(\nabla_X \phi)Y = \alpha[g(X, Y)\xi - \epsilon \eta(Y)X] + \beta[g(\phi X, Y)\xi - \delta \eta(Y)\phi X], \quad (2.7)$$

holds for some smooth functions  $\alpha$  and  $\beta$  on  $M$  and  $\epsilon = \pm 1$ ,  $\delta = \pm 1$ . For  $\beta = 0, \alpha = 1$ , an  $(\epsilon, \delta)$ -trans-Sasakian manifold reduces to an  $(\epsilon)$ -Sasakian and for  $\alpha = 0, \beta = 1$  it reduces to a  $(\delta)$ -Kenmotsu manifold.

Let  $(M, g)$  be a  $(\epsilon, \delta)$ -trans-Sasakian manifold. Then from (2.7), it is easy to see that

$$(\nabla_X \xi) = -\epsilon \alpha \phi X - \beta \delta \phi^2 X, \quad (2.8)$$

$$(\nabla_X \eta)Y = -\alpha g(Y, \phi X) + \epsilon \delta \beta g(\phi X, \phi Y), \quad (2.9)$$

$$\xi \alpha + 2\alpha \beta = 0. \quad (2.10)$$

In a 3-dimensional  $(\epsilon, \delta)$ -trans-Sasakian manifold, the curvature tensor  $R$  and Ricci tensor  $S$  are given by[5]

$$\begin{aligned} R(X, Y)Z = & (2A - \frac{r}{2})(g(Y, Z)X - g(X, Z)Y) + B(g(Y, Z)\eta(X) - g(X, Z)\eta(Y))\xi \\ & + B\eta(Z)(\eta(Y)X - \eta(X)Y), \end{aligned} \quad (2.11)$$

$$R(X, Y)\xi = \epsilon[(\alpha^2 - \beta^2) + (\frac{2 - \epsilon}{2})r][\eta(Y)X - \eta(X)Y], \quad (2.12)$$

$$S(X, Y) = Ag(X, Y) + B\eta(X)\eta(Y), \quad (2.13)$$

where  $\epsilon\delta = 1$ ,  $A = (\frac{r}{2} - (\alpha^2 - \beta^2))$ ,  $B = (3(\alpha^2 - \beta^2) - \epsilon\frac{r}{2})$  and  $r$  is the scalar curvature.

Let  $(M, g)$  be a Riemannian manifold with metric  $g$ . The metric  $g$  is called a Ricci soliton if [8]

$$L_V g + 2S + 2\lambda g = 0, \quad (2.14)$$

where  $L$  is the Lie derivative,  $S$  is a Ricci tensor,  $V$  is a complete vector field on  $M$  and  $\lambda$  is a constant. So Ricci soliton is a generalization of Einstein metric. In [9], Ramesh sharma started the study of the Ricci solitons in contact geometry. Later Mukutmani Tripathi [11], Cornelia Livia Bejan and Mircea Crasmareanu [2] and others extensively studied Ricci solitons in contact metric manifolds. The Ricci soliton is said to be shrinking, steady and expanding according as  $\lambda$  is positive, zero and negative respectively.

We also study  $K$ -torse forming vector fields in 3-dimensional  $(\epsilon, \delta)$ -trans-Sasakian manifold. Torse forming vector fields were introduced by K.Yano [6],

$$\nabla_X \rho = aX + \pi(X)\rho, \quad (2.15)$$

where  $\rho$  is a vector field and  $\pi$  is a non-zero 1-form. Further, a complex analogue of a torse forming vector field is called  $K$ -torse forming vector field and it was introduced by S.Yamaguchi [10],

$$\nabla_X \rho = aX + b\varphi X + B(X)\rho + D(X)\varphi\rho, \quad (2.16)$$

where  $\rho$  is a vector field,  $a$  and  $b$  are functions,  $B(X)$  and  $D(X)$  are 1-forms.

A 3-dimensional  $(\epsilon, \delta)$ -trans-Sasakian manifold  $(M, g)$  is called generalized recurrent [12], if its curvature tensor  $R$  satisfies the condition

$$(\nabla_X R)(Y, Z)(W) = A(X)R(Y, Z)W + B(X)[g(Z, W)Y - g(Y, W)Z], \quad (2.17)$$

where  $A$  and  $B$  are two 1-forms and  $B$  is non zero.

### 3. Ricci Soliton

Let  $M$  be a 3-dimensional  $(\epsilon, \delta)$ -trans-Sasakian manifold with metric  $g$ . A Ricci soliton is a generalization of an Einstein metric and defined on a Riemannian manifold  $(M, g)$  by (2.14).

Let  $V$  be pointwise collinear with  $\xi$  ie.,  $V = b\xi$ , Then (2.14) implies

$$g(\nabla_X b\xi, Y) + g(\nabla_Y b\xi, X) + 2S(X, Y) + 2\lambda g(X, Y) = 0, \quad (3.1)$$

which is reduced to

$$bg(\nabla_X \xi, Y) + (Xb)\eta(Y) + bg(\nabla_Y \xi, X) + (Yb)\eta(X) + 2S(X, Y) + 2\lambda g(X, Y) = 0. \quad (3.2)$$

Using (2.8) in (3.2), we obtain

$$2b\beta\delta g(X, Y) - 2b\beta\eta(X)\eta(Y) + (Xb)\eta(Y) + (Yb)\eta(X) + 2S(X, Y) + 2\lambda g(X, Y) = 0. \quad (3.3)$$

Replace  $Y$  by  $\xi$  in (3.3) and use (2.13) to get

$$2b\beta\delta\epsilon\eta(X) - 2b\beta\eta(X) + (Xb) + (\xi b)\eta(X) + 2A\epsilon\eta(X) + 2B\eta(X) + 2\lambda\epsilon\eta(X) = 0. \quad (3.4)$$

Again putting  $X = \xi$  in (3.4), we obtain

$$\xi b = -b\beta\delta\epsilon + b\beta - A\epsilon - B - \lambda\epsilon. \quad (3.5)$$

By using (3.5) in (3.4), we get

$$Xb = [-b\beta\delta\epsilon + b\beta - A\epsilon - B - \lambda\epsilon]\eta(X), \quad (3.6)$$

or

$$db = [-b\beta\delta\epsilon + b\beta - A\epsilon - B - \lambda\epsilon]\eta, \quad (3.7)$$

where we have put  $\nabla_X = d$ .

Applying  $d$  on (3.7), we get

$$[b\beta\delta\epsilon - b\beta + A\epsilon + B + \lambda\epsilon]d\eta = 0. \quad (3.8)$$

Since  $d\eta \neq 0$  we have

$$b\beta\delta\epsilon - b\beta + A\epsilon + B + \lambda\epsilon = 0. \quad (3.9)$$

Using (3.9) in (3.7) then  $db = 0$  and hence  $b$  is a constant. Therefore from (3.3) it follows

$$S(X, Y) = -(\lambda + b\beta\delta)g(X, Y) + b\beta\eta(X)\eta(Y), \quad (3.10)$$

which implies that  $M$  is of constant scalar curvature provided  $\beta = \text{constant}$ . This leads to the following:

**Theorem 3.1.** *Let  $M$  be a 3-dimensional  $(\epsilon, \delta)$ -trans-Sasakian manifold in which the metric tensor  $g$  is Ricci soliton with the vector field  $V$  as pointwise collinear with  $\xi$ , then  $M$  is a space of constant scalar curvature provided  $\beta$  is a constant.*

Now let  $V = \xi$ . Then the (2.14) becomes

$$L_\xi g + 2S + 2\lambda g = 0. \quad (3.11)$$

Using (2.8), we get

$$L_\xi g(X, Y) = 2\beta\delta g(X, Y) - 2\beta\delta\epsilon\eta(X)\eta(Y), \quad (3.12)$$

and therefore,

$$\begin{aligned} (L_\xi g + 2S)(X, Y) &= 2\beta\delta[g(X, Y) - \epsilon\eta(X)\eta(Y)] - 2\left[\left(\frac{r}{2} - (\alpha^2 - \beta^2)\right)g(X, Y) \right. \\ &\quad \left. + (3(\alpha^2 - \beta^2) - \epsilon\frac{r}{2})\eta(X)\eta(Y)\right]. \end{aligned} \quad (3.13)$$

Using (3.13) in (3.11), we obtain

$$2\left[-\frac{r}{2} + \beta\delta + (\alpha^2 - \beta^2) + \lambda\right]g(X, Y) - 2\left[\beta\delta\epsilon + 3(\alpha^2 - \beta^2) - \epsilon\frac{r}{2}\right]\eta(X)\eta(Y) = 0. \quad (3.14)$$

Take  $X = Y = \xi$  in (3.14). We get

$$\lambda = \frac{(\beta^2 - \alpha^2)(\epsilon + 3) - r(\epsilon - 1)}{2\epsilon}. \quad (3.15)$$

Hence we state the following

**Theorem 3.2.** *Let  $M$  be a 3-dimensional  $(\epsilon, \delta)$ -trans-Sasakian manifold. Then a Ricci soliton  $(g, \xi, \lambda)$  in  $(M, g)$  is:*

- (i) *shrinking for  $(\beta^2 - \alpha^2)(\epsilon + 3) > r(\epsilon - 1)$ ,*
- (ii) *expanding for  $(\beta^2 - \alpha^2)(\epsilon + 3) < r(\epsilon - 1)$ .*

**Corollary 3.1.** *In a 3-dimensional  $(\epsilon, \delta)$ -trans-Sasakian manifold  $M$  Ricci soliton is:*

- (i) *shrinking if  $\beta > \alpha$  and  $\epsilon = 1$ ,*
- (ii) *expanding if  $\beta < \alpha$  and  $\epsilon = 1$ ,*
- (iii) *steady if  $\alpha = \beta$  and  $\epsilon = 1$ .*

Suppose  $(M, g)$  is a 3-dimensional  $(\epsilon, \delta)$ -trans-Sasakian manifold and  $(g, V, \lambda)$  is a Ricci soliton in  $(M, g)$ . If  $V$  is a conformal killing vector field, then

$$L_V g = \rho g, \quad (3.16)$$

for some scalar function  $\rho$ . From (2.14)

$$S(X, Y) = -[\lambda g(X, Y) + \frac{1}{2}L_V g(X, Y)]. \quad (3.17)$$

Now from (3.16) it follows that

$$S(X, Y) = -(\lambda + \frac{\rho}{2})g(X, Y), \quad (3.18)$$

$$QX = -(\lambda + \frac{\rho}{2})X, \quad (3.19)$$

$$r = -3(\lambda + \frac{\rho}{2}). \quad (3.20)$$

Put  $X = Z = \xi$  and use (3.20) in (2.11) to obtain

$$R(\xi, Y)\xi = (\alpha^2 - \beta^2)(3 - 2\varepsilon)[\eta(Y)\xi - Y]. \quad (3.21)$$

Put  $X = \xi$  in (2.12) to obtain

$$R(\xi, Y)\xi = \varepsilon \left[ (\alpha^2 - \beta^2) + \left( \frac{2 - \varepsilon}{2} \right) r \right] [\eta(Y)\xi - Y]. \quad (3.22)$$

Comparing (3.21) and (3.22), we obtain

$$r = \frac{6(1 - \varepsilon)(\alpha^2 - \beta^2)}{(2\varepsilon - 1)}. \quad (3.23)$$

By using (3.20) and (3.23), we get

$$\lambda = \frac{2(\varepsilon - 1)(\alpha^2 - \beta^2)}{(2\varepsilon - 1)} - \frac{\rho}{2}. \quad (3.24)$$

**Theorem 3.3.** *Let  $M$  be a 3-dimensional  $(\varepsilon, \delta)$ -trans-Sasakian manifold admitting Ricci soliton  $(g, V, \lambda)$ , where  $V$  is conformal killing vector field. Then  $(g, V, \lambda)$  is:*

- i) expanding for  $\rho > \frac{4(\varepsilon-1)(\alpha^2-\beta^2)}{(2\varepsilon-1)}$ ,
- ii) shrinking for  $\rho < \frac{4(\varepsilon-1)(\alpha^2-\beta^2)}{(2\varepsilon-1)}$
- iii) steady for  $\rho = \frac{4(\varepsilon-1)(\alpha^2-\beta^2)}{(2\varepsilon-1)}$ .

#### 4. $K$ -Torse Forming Vector Field

In 3-dimensional  $(\epsilon, \delta)$ -trans-Sasakian manifold,  $\xi$  is always a  $K$ -torse forming vector field. Take  $\rho = \xi$  in (2.16) and compare the result with (2.8) to obtain  $a = \beta\delta$ ,  $b = -\epsilon\alpha$ ,  $B(X) = -\beta\eta(X)$  and  $D(X)=0$ . Then it implies

$$\nabla_X \xi = \beta\delta X - \epsilon\alpha\varphi X - \beta\eta(X)\xi, \quad (4.1)$$

$$(\nabla_X \eta)Y = \beta\delta g(X, Y) - \epsilon\alpha g(\varphi X, Y) - \epsilon\beta\eta(X)\eta(Y), \quad (4.2)$$

$$R(X, Y)\xi = (a\beta + b\alpha)[\eta(X)Y - \eta(Y)X], \quad (4.3)$$

$$R(\xi, X)Y = (a\beta + b\alpha)[\eta(Y)X - \epsilon g(X, Y)\xi], \quad (4.4)$$

and

$$S(Y, \xi) = -2(a\beta + b\alpha)\eta(Y), \quad (4.5)$$

where  $\alpha$  and  $\beta$  are constants.

Taking  $Y = W = \xi$  in (2.17), we obtain

$$(\nabla_X R)(\xi, Z)(\xi) = A(X)R(\xi, Z)\xi + B(X)[\epsilon\eta(Z)\xi - \epsilon Z]. \quad (4.6)$$

By the definition of covariant derivative, we have

$$(\nabla_X R)(\xi, Z)(\xi) = \nabla_X R(\xi, Z)\xi - R(\nabla_X \xi, Z)\xi - R(\xi, \nabla_X Z)\xi - R(\xi, Z)\nabla_X \xi. \quad (4.7)$$

Using (4.1), (4.2) and (4.4) in (4.7), we get

$$\begin{aligned} (\nabla_X R)(\xi, Z)(\xi) &= d(a\beta + b\alpha)(X)[Z - \eta(Z)\xi] + (a\beta + b\alpha)[(1 - \delta)\beta g(X, Z)\xi \\ &\quad + (\epsilon - 1)\alpha g(\varphi X, Z)\xi + (\epsilon - 1)\beta\eta(X)\eta(Z)\xi + 2\eta(\nabla_X Z)\xi - 2\beta\eta(X)Z + 2\epsilon\beta\eta(X)Z]. \end{aligned} \quad (4.8)$$

From (4.6) and (4.8), we have

$$\begin{aligned} &d(a\beta + b\alpha)(X)[Z - \eta(Z)\xi] + (a\beta + b\alpha)[(1 - \delta)\beta g(X, Z)\xi + (\epsilon - 1)\alpha g(\varphi X, Z)\xi \\ &\quad + (\epsilon - 1)\beta\eta(X)\eta(Z)\xi + 2\eta(\nabla_X Z)\xi - 2\beta\eta(X)Z + 2\epsilon\beta\eta(X)Z] \\ &= A(X)[(a\beta + b\alpha)(Z - \eta(Z)\xi)] + B(X)[\epsilon\eta(Z)\xi - \epsilon Z]. \end{aligned} \quad (4.9)$$

Put  $Z = \xi$  and use (2.2), (2.4), (2.5) in (4.9) to obtain

$$(a\beta + b\alpha)[2(\epsilon - 1)\beta\eta(X)\xi + \eta(\nabla_X \xi)\xi] = 0. \quad (4.10)$$

If  $(a\beta + b\alpha) \neq 0$  and  $\epsilon = 1$ , then (4.10) is

$$\nabla_X \xi = 0. \quad (4.11)$$

Thus we have

**Theorem 4.4.** *In a generalized recurrent 3-dimensional  $(\varepsilon, \delta)$ -trans-Sasakian manifold  $M$  the vector field  $\xi$  is co-symplectic provided  $(a\beta + b\alpha) \neq 0$  and  $\varepsilon = 1$ .*

Suppose the Ricci tensor  $S$  is semi conjugate in 3-dimensional  $(\varepsilon, \delta)$ -trans-Sasakian manifold  $M$  with respect to the vector field  $\xi$ , which is  $K$ -torse forming vector field in  $M$ . So  $R(X, \xi).S(Y, Z) = 0$ .

Then we have

$$S(R(X, \xi)Y, Z) + S(Y, R(X, \xi)Z) = 0. \quad (4.12)$$

Using (4.4) and (4.5) in (4.12), we get

$$(a\beta + b\alpha)[S(X, Z)\eta(Y) + 2\varepsilon(a\beta + b\alpha)g(Y, X)\eta(Z) + S(Y, X)\eta(Z) + 2\varepsilon(a\beta + b\alpha)g(X, Z)\eta(Y)] = 0. \quad (4.13)$$

Put  $Z = \xi$  and use (2.5),(4.5) in (4.13) to obtain

$$S(X, Y) = -2\varepsilon(a\beta + b\alpha)g(X, Y), \quad (4.14)$$

which gives

$$r = -6\varepsilon(a\beta + b\alpha), \quad (4.15)$$

where  $a$  and  $b$  are constants. Hence  $r$  is constant.

**Theorem 4.5.** *Suppose the ricci tensor  $S$  in a 3-dimensional  $(\varepsilon, \delta)$ -trans-Sasakian manifold  $M$  is semi-conjugate with respect the vector field  $\xi$ . Then  $M$  is a space of constant scalar curvature.*

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